Great Mathematicians and the Geosciences: From Leibniz and Newton to Einstein and Hilbert

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Presented to the Leibniz Society of Sciences at Berlin
November 14, 2008

With a contribution *Mathematics and Biology* (2009)

by

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Abstract

The paper consists of three interrelated parts.

A. Classical Physics.
Beginning with Newton, physics and mathematics are inseparably connected and have grown together. That, according to Newton, the fall of an apple and the orbit of the Moon around the Earth are based on the same natural laws, is still remarkable even today. For eminent mathematicians such as Laplace, Lagrange and Gauss, the natural sciences have not only been basic for explaining and describing the gravitational field of the Earth, but they have received also purely mathematical inspirations from the geosciences.
B. Complexity.
A standard recent example is chaos theory, or the theory of nonlinear dynamic systems, founded by Henri Poincaré around 1890 in his research of celestial mechanics and made popular, among others, by the fundamental treatment of weather forecasting by E. Lorenz (1963). Generally one speaks of Complexity Theory with possible applications to geodynamics, biology, and medicine. Particulary interesting is a general mathematical theory of biology, which is being developed by G.J.Chaitin. The beginning of Chaitin's paper are reproduced here by kind permission of the author.
C. Inverse problems.
A first important example is Gauss' least-squares adjustment of overdetermined systems. Nowadays, there are methods which are of increasing importance for geodesy, geophysics, medicine, etc. Some examples are the determination of masses from gravimetry, and seismic tomography, which has a similar mathematical structure as Magnetic Resonance Tomography in medicine.

Zusammenfassung

Die Arbeit besteht aus drei zusammenhängenden Teilen.

A. Klassische Physik.
Seit Newton sind die Physik und die Mathematik untrennbar verbunden und miteinander gewachsen. Dass nach Newton der Fall eines Apfels vom Baum und die Bahn des Mondes um die Erde den gleichen Naturgesetzen unterliegen, ist auch heute noch beachtlich. Für große Mathematiker wie Laplace, Lagrange und Gauß sind die Naturwissenschaften nicht nur grundlegend für die Erklärung und Beschreibung des Erdschwerefeldes gewesen, sondern sie haben dadurch auch rein mathematische Anregungen bekommen.
B. Komplexität.
Ein Standardbeispiel ist die Chaostheorie, oder Theorie der nichtlinearen dynamischen Systeme. Sie wurde Henri Poincaré um 1890 durch seine Arbeiten in der Himmelsmechanik begründet. Heute ist sie bekannt vor allem durch die fundamentalen Untersuchungen über Wettervorhersage von E. Lorenz (1963). Allgemein spricht man von *Komplexitätstheorie* mit möglichen Anwendungen in Geodynamik, Biologie und Medizin. Besonders interessant ist ein Beitrag von G.J. Chaitin mit einem allgemeinen Ansatz für eine mathematische Theorie der Biologie, dessen Anfang mit freundlicher Genehmigung des Verfassers wiedergegeben wird.
C. Inverse Probleme.
Ei erstes wichtiges Beispiel ist die Ausgleichung überbestimmter Systeme nach kleinsten Quadraten von Gauß. Heute gibt es Methoden, deren Schwierigkeit und Wichtigkeit für die Praxis (von Massenbestimmung aus Gravitation bis zur seismischen und Magnetresonanz-Tomographie) viele zeitgenössischen Mathematiker für Geodäsie, Geophysik und Medizin interessiert.

A. Classical Physics

1. Introduction

Beginning with Newton, physics and mathematics are inseparably connected and have grown together. (This is not the case for all natural sciences, for instance biology.) The fact that according to Newton, the fall of an apple and the orbit of the Moon around the Earth are based on the same natural laws, is remarkable even today. The fact that astronomy as an *explaining* science is subject to the same physical laws as geodesy describing the earth and its gravity field, would not have been understandable to the great practical astronomers and land surveyors of the near East some thousand years ago.

For the eminent mathematicians beginning with Newton, the natural sciences have not only been basic for explaining and describing the earth, but mathematicians have received from them also purely mathematical inspirations.

I know that mathematics is not always liked by other sciences. Thus I shall try to keep my addiction to mathematics within limits, so that our colleagues from other disciplines get an overview that is hopefully generally understandable. If the terminology is getting too special, skip the terms and look at the figures. (There will be hardly any formula!). (A detailed review of many relevant problems is found in the book „Science, Mind and the Universe“ (in the present text abbreviated as SMU), which can be freely copied from the PDF-version in the present web page http://www.helmut-moritz.at.)

We understand by classical physics everything treated by a standard textbook on theoretical physics such as Landau-Lifschitz or the Feynman Lectures on Physics, thus besides theoretical mechanics also relativity and quantum theory. Now we shall sketch a few geodetic applications.

2. Physics and Geodesy

In the 18th century, geodesy was a main area of work of the great French mathematicians. Around 1740, the French Academy of Sciences sent “*grade measuring*” expeditions to Peru and Lapland in order to determine the size of the Earth ellipsoid. The results were intended to serve for the determination of the meter and hence the metric system. At that time, geodesy was indeed an important basic science.

Thus, mathematicians such as Lagrange, Laplace, Clairaut and Maupertuis (who also led the Lapland expedition) worked out the corresponding theories of the figure of the Earth and its gravity field. We mention here only the fundamental work done mainly by Laplace and Lagrange on *spherical harmonics,* which showed their basic practical importance especially today, when we can determine them by means of artificial Earth satellites to a high precision. (For this, we of course need new highly accurate theories and methods of observation and computation.)

An indication of the relevance of geodesy to the general atmosphere of thinking in these times is the fact that the great writer of science fiction, Jules Verne, travelled in his novels not only from the Earth to the Moon but also to the center of the Earth. He even wrote a geodetic novel “Trois Russes and trois Anglais”. This work describes a grade measurement from North to South through the whole African continent. The constellation of the team anticipates international cooperation, with its successes and its tensions. (By the way, such a grade measurement in Africa was indeed performed by the Americans around 1956!)

In Germany, Carl Friedrich Gauss developed the least-squares method and used it for the adjustment of geodetic measuring errors. (In fact, however, it had a much wider scope of application throughout mathematics, besides being the first “ill-posed problem”; see secs. 9 and 10.) Gauss also discovered the intrinsic differential geometry of curved surfaces, likewise inspired by his geodetic work. It lead Riemann to the differential geometry of higher-dimensional curved spaces, which Einstein and Hilbert independently used to find a general theory of space, time and gravitation (around 1915). This “General Theory of Relativity” is necessary for a highly precise determination of satellite orbits, for instance with the Global Positional System (GPS). (So to speak, Einstein sits as a blind passenger in every automobile guided by GPS.)

Hilbert’s name is connected primarily with the second great physical theory of the 20^{th} century. The Quantum Theory in its most perfect form was developed by Heisenberg and Schrödinger, which is based on infinite-dimensional Hilbert space. (If we complain that such a space cannot be visualized, we should keep in mind that, at present, computers solving a system of linear equation containing 100000 unknowns, mathematically work in a space of 100000 dimensions, which cannot visualized either!)

Gauss’ method of least squares in Hilbert space has found an important geodetic application, namely *least-squares collocation*, used, for instance, for the precise determination of the above-mentioned spherical harmonics of the gravitational field of the Earth. This field is needed today for the computation and use of satellite orbits.

We must, however, also point out that the most important and influential thinker in physical geodesy after 1945 was the Russian geodesist M. S. Molodensky. We will come back to him in Section 10.

B.Complexity

3. From Classical Mechanics to Chaos Theory

A completely new area of mathematics has been popularly known as Chaos Theory or abstractly, *Theory of nonlinear dynamical systems* . Synonyms are fractals or deterministic chaos. It is not so much a clear-cut term as a set of related mathematical theories. Later we shall understand this better.

Newtonian (and also Einsteinian) “classical” celestial mechanics were believed to be rigorously deterministic (SMU p. 73 ff.). However, Henri Poincaré showed around 1890, that orbits of celestial bodies may become “chaotic”. A standard picture (Fig. 1) indeed shows strange 3D-orbits as formed by their intersection with the plane of the picture. I do not hesitate to regard the great French mathematician Henri Poincaré as *the founder of chaos theory*. His book “Nouvelles méthodes de la mécanique céleste” is considered the first mathematical description of the chaotic behaviour of nonlinear dynamical systems and perhaps the greatest progress in celestial mechanics since

Newton[1]. He also fully understood the meteorological weather prediction problem (SMU, pp. 80-82; a beautiful quotation is found on p. 243 of SMU), as well as the mechanics of dice-throwing leading to equal probability of all the faces if the die is symmetric (SMU, pp. 83-84). He must have had in his mind a comprehensive nontrivial synthesis between classical mechanics (Laplace's determinism), instability, chaos, and probability, as shown by his great popular books.[2]

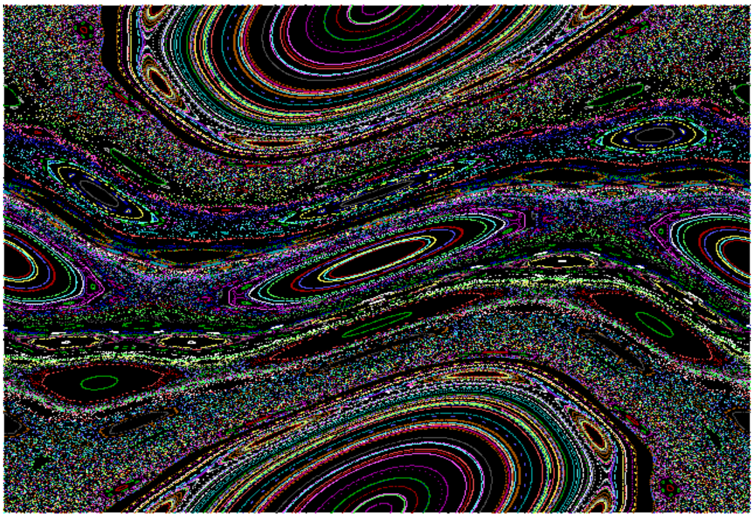

Fig.1. Poincaré.gif

4. Chaos in the Geosciences

I shall consider here some areas among which the first has already been touched before.
(1) *Meteorology.* Well known is *Lorenz' butterfly*, which goes back to E. Lorenz (1963) and illustrates the instability and principal uncertainty of weather prediction (Fig. 2).
The neighbouring trajectories demonstrate two possible weather forecasts for the same day, found in two different newspapers. Lorenz has also written a beautiful general book[3].
(2) *Geomagnetism.* Even the difficult theory of the electromagnetic *geodynamo* is impossible without chaos theory.
(3) *Predictions of earthquakes* or of the occurring of avalanches. In principle, the problems are somewhat similar to the meteorology problem, but the uncertainty is much greater[4].

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1 S. Fröba and A. Wassermann, *Die bedeutendsten Mathematiker,* Matrix Verlag, Wiesbaden, 2007, p.192.

2 When I was a young professor at the Berlin Technical University, I studied Poincaré around 1970 and discovered what to me appeared as a most wonderful secret geodetic link between Berlin and Paris. The well-known definition of geodesy as the science of the figure of the Earth and its gravitational field, crystallized most concisely around 1870 by combining the thinking of two scientists of the Potsdam Geodetic Institute, Friedrich Robert Helmert and Heinrich Bruns. At about the same time, Heinrich Bruns published a small but deep and influential article on the convergence problem of the series of celestial mechanics (Bemerkungen zur Theorie der allgemeinen Störungen, *Astron. Nachrichten 109*, 216-222, 1884) which foreshadows Poincaré' great work. This may be only a coincidence but to me it indicates the fact, dear to me, that scientific ideas transcend political borders. (Helmert and Poincaré were also active in the International Geodetic Association!)

3 E. Lorenz, *The Essence of Chaos,* Univ. of Washington Press, 1993.

4 D. L. Turcotte, *Fractals and Chaos in Geology and Geophysics,* 2nd ed., Cambridge Univ. Press, 1997.

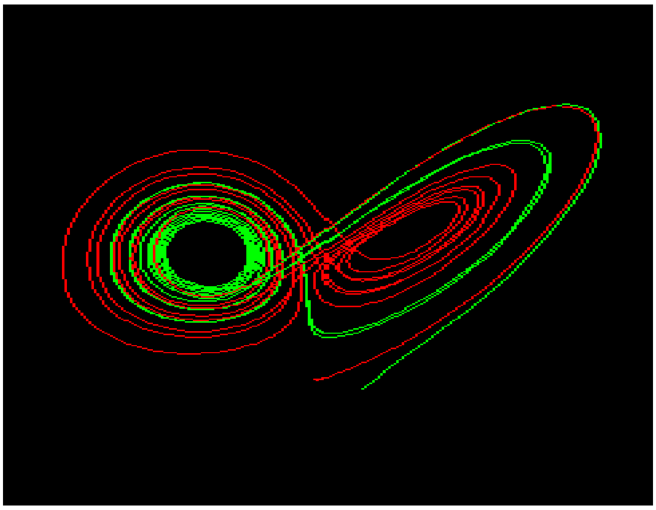

Fig.2. Lorenz.gif

(4) *Fractal Surfaces* are a means of representation and interpolation in high mountains (Fig. 3).

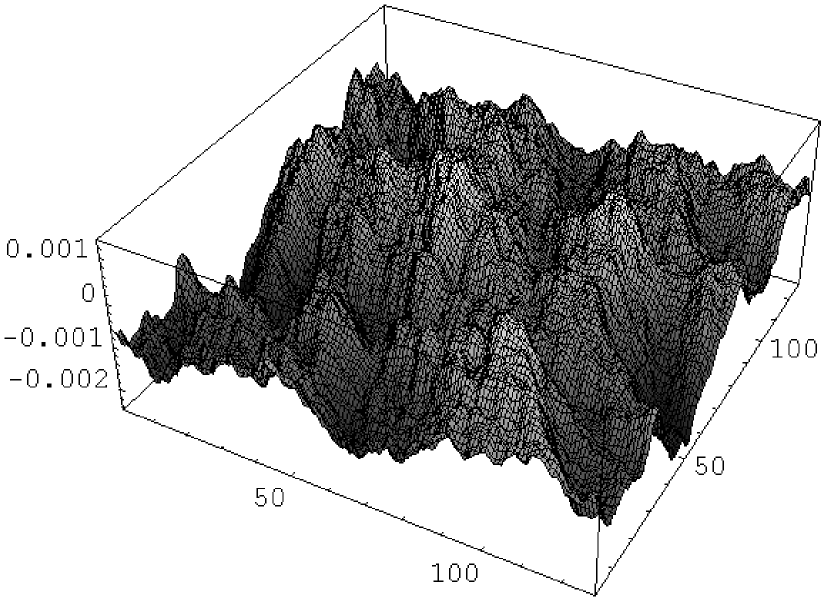

Fig.3. Fractal high mountains

It is obvious that such frightening images occur in the movie industry rather than in reality[5].

5. Leibniz, Mandelbrot and Fractals

In his great book *The Fractal Geometry of Nature* (Freeman, New York, 1977), Benoit Mandelbrot gives a comprehensive presentation of fractals, which has fascinated a

[5] H.O. Peitgen and D, Saupe (eds.), *The Science of Fractal Images*, Springer, 1998.

generation of mathematicians and applied scientists, but also young computer fans. The Mandelbrot fractal (Fig. 4 below) and the Lorenz butterfly (Fig. 2), have become the logos of the new chaos theory. The Mandelbrot fractal consists of a main „apple“,

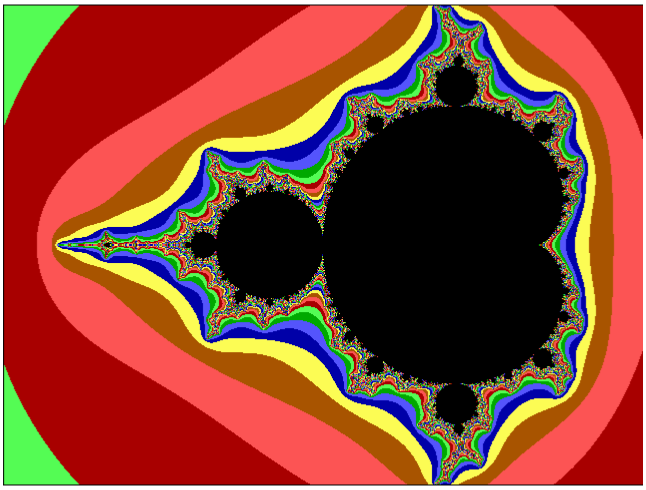

Fig.4. The Mandelbrot fractal

surrounded by smaller and smaller„self-similar“ “applets” (in black). Such figures permit an intuitive approach to chaos theory. For the “moderate” mathematician: they are non-differentiable, but nevertheless regular curves or structures, which are throughout “broken” (in Latin *fractus*)**.** This is a crude and non-rigorous intuitive description. For the rigorous mathematician there are exact definitions of any degree of difficulty.

Interestingly enough, the genesis of a fractal is similar to the logical sharpening of Leibniz’ concept of differential in the last few centuries by Robinson and others[6]. According to Leibniz, the derivative of a function $y = f(x)$ is a genuine quotient dy/dx of two differentials dx and dy which are “infinitesimally small” but not zero. They are extremely well suited for computation and have always been used, although despised by rigorous mathematicians of the past. Modern logic has completely rehabilitated the infinitesimal differentials. His ingenious intuition has been right.

Mandelbrot, a great admirer of Leibniz, has alluded the relation with fractals in several places of his book and tried to unify the antithesis between differentials (smooth curves) and fractals (“broken” curves) to a synthesis (his book, §41).

Let us also mention that Leibniz, that universal genius, has also occupied himself with the structure of the Earth’s interior[7].

6. In Nature there are No Straight Lines: Fractals in Biology and Medicine

Already in the beginning of modern fractal theory around 1960 it was recognized that the Euclidian-Cartesian geometry does not completely fit to a nature untouched by Man. In nature there are hardly any straight lines (apart from light rays), at least not in biology.

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[6] D. Laugwitz, Infinitesimalkalkül. B.I. Verlag. 1978.

[7] W. Kertz, *Geschichte der Geophysik*, Georg Olms Verlag 1999 (Chapter 1, sec.3). The author is indebted to Prof. Heinz Kautzleben for this remark.

Hardly a tree is as perfectly straight as the edge of a house. In mineralogy, the crystal edges are pretty straight, but this is noted mainly by mineralogists. If the brain surgeon penetrates into the human brain, he may sometimes use a scalpel with a perfectly straight blade, but the brain windings are far from straight. Of course, on can imbed the brain into a Cartesian net, but this is done by man and not by nature. A nice modelling of lung tissue is the three-dimensional fractal by Sierpinski-Menger (Fig.5), somewhat similar to a sponge (the pieces of straight lines are only due to an imperfect approximation; in reality there are only points):

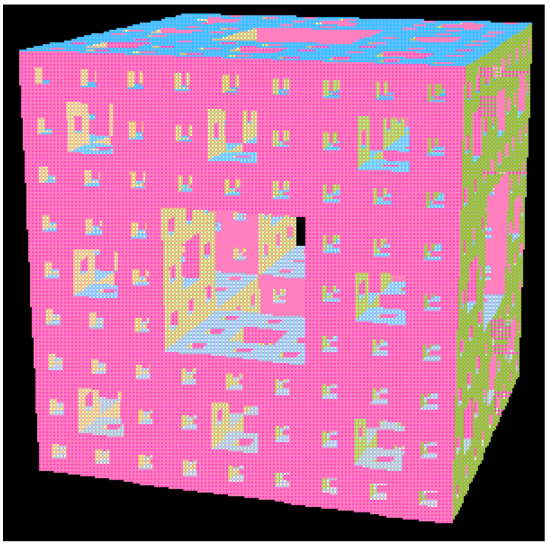
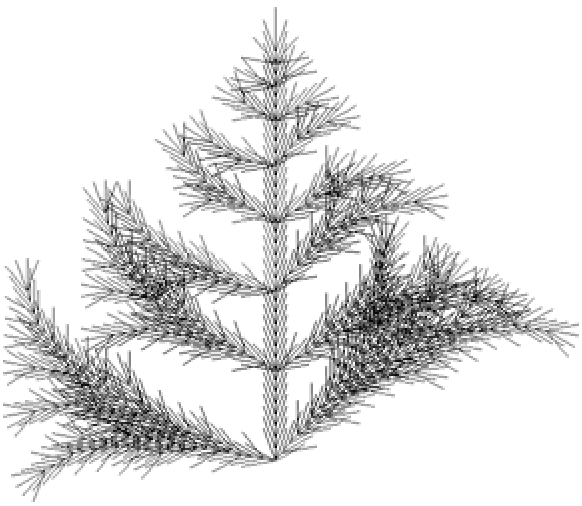


Fig.5. menger.gif (after R. Maeder)



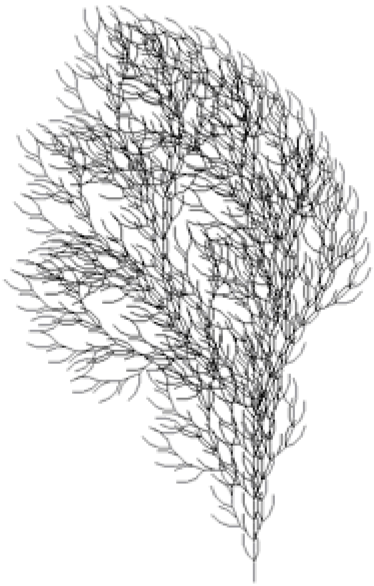


Fig.6. Fractal trees (after P. Bourke)

There are also beautiful fractal trees (Fig. 6). No wonder that biologists and medical scientists have immediately tried to apply chaos theory.[8] *There are simply no straight lines in a nature unaffected by man.*

7. Facing Complexity

In chaos theory there are important discoverers. As we have seen in Section 3, Poincaré was a mathematical genius who first showed that even classical celestial mechanics is not as simple and stable as Laplace thought. Thus Poincaré discovered nonlinear mechanics. As we have already mentioned, Poincaré investigated also the relation between instability and probability. It is evident that *probability theory* is related to chaos and complexity: think of the "probability of rain" in a weather forecast.

E. Lorenz revolutionized meteorology. Mandelbrot is a comprehensive genius, who discovered fractal structures all over nature.

In this paper, we cannot show any more fractal structures. Many nice pictures can be found in the internet in so-called "fractal galleries".
For books see, e.g., Footnote[9].

About terminology: one speaks of

- Nonlinear dynamical systems (mathematics of high level)
- Chaos theory (instability, meteorology)
- „Deterministic chaos"
- Fractal geometry (Mandelbrot)

All these concepts are in general pretty much related, but the connection is not completely seamless. The most important time of chaos theory may have been 1980-2000. At present there seems to be a certain stagnation in chaos theory. There is no lack of unknown problems, particular in geo- and biosciences. Perhaps there comes a new Einstein who, in a change of paradigms according to Kuhn (SMU p.155), creates a new synthesis, which is usually referred to as *complexity theory*.

Intuitively we may speak of *complexity* if the system has a great number of elements combined with a rich structure. A heap of sand will hardly be considered a complex system; an ant-hill will already be a better candidate, and a living organism is certainly a typical example. Briefly we may say that complexity is a synthesis of order and chaos, or that *complexity lies at the edge of order and chaos* (M. M. Waldrop). Known all over the world is the *Santa Fe Institute of Complexity.*

The very geo- and biosciences would require a unified theory of complexity. Into such a theory also algorithmic complexity (W. I. Kolmogorov, G. J. Chaitin) would have to fit. For more details see SMU, p. 170-172. The question is whether such a *unified* theory of complexity is possible, rather than a set of more or less connected theories. However, even mathematics itself is not a unified deductive system, but rather a set or collection of systems as different as number theory and differential geometry.

A wonderfully clear and enlightening account of ***questions*** posed by biology (especially evolution) to mathematics (complexity theory, metamathematics, entropy, computer science), has been given by Gregory Chaitin in a paper put on internet (2009), of which the first part will be presented now, by kind permission of the author.

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[8] A classic is L.Glass and M. C. Mackey, *From Clocks to Chaos: the Rhythms of Life,* Princeton Univ. Press, 1988.
[9] Readable and important books are J. Gleick, *Chaos :Making a New Science*, Cardinal-Penguin, London, 1988, and on a particular high level, I. Stewart, *Does God Play Dice?*, Penguin Books, 1990.

8. Mathematics and Biology

by G. J. Chaitin, *IBM Research*

Chaitin writes:

"Abstract

It would be nice to have a mathematical understanding of basic biological concepts and to be able to prove that life must evolve in very general circumstances. At present we are far from being able to do this. But I'll discuss some partial steps in this direction plus what I regard as a possible future line of attack.

Introduction: Goals of Our Theory

- 200th anniversary of Darwin's birth, 150th anniversary of *The Origin of Species.*
- The unreasonable effectiveness of mathematics in physics (Wigner) **versus** The lack of effectiveness of mathematics in biology (Gelfand).
- We wish to extract the fundamental mathematical ideas contained in biology.
- We wish to prove theorems about extremely simple unrealistic models, not run simulations of extremely complicated realistic models.
- Our goal is not to realistically simulate biological evolution, but to represent mathematically the fundamental biological principles of evolution in such a manner that we can prove that evolution must take place.
- This may be regarded as a toy model, but we do not see it as a toy, we see it as a way to eliminate inessential distractions that only serve to confuse the issues!
- Theories are lies that help us to see the truth (Picasso).
- Math is extremely single-minded and can only deal successfully with a single idea at a time
- If Darwin's theory of evolution is as fundamental, basic and general as most biologists think, then it must be possible to extract some basic mathematical ideas from it.
- Nothing makes sense in biology except in the light of evolution (Dobzhansky).
- It is scandalous that we do not have a mathematical proof that evolution works!
- I am a pure mathematician, not a biologist: I am trying to find the Platonic ideal of evolution, the archetypical behavior, not the messy version that takes place in the real world!
- The aim is proofs, not realistic simulations.
- Another way our model differs from previous models: Our goal is to understand biological creativity and the major transitions in evolution, not gradual changes."

(End of Chaitin's contribution, quoted by permission of the author; for a continuation see his original paper www.umcs.maine.edu/~chaitin/jack.html .)

C. Inverse Problems

9. Potential Theory

At the beginning of any textbook in physical geodesy[10] we have the simple formula for the gravitational potential *V* of a body such as the Earth (Fig. 7).

$$V = \iiint \frac{1}{\ell} \rho \, dv$$

Here ρ denotes the density of the masses inside the Earth, *dv* is the volume element, ℓ is the distance from a space point *P* to *dv*, and the triple integral is extended over the total volume of the Earth (Fig. 7) (the gravitational constant has been put equal to 1).

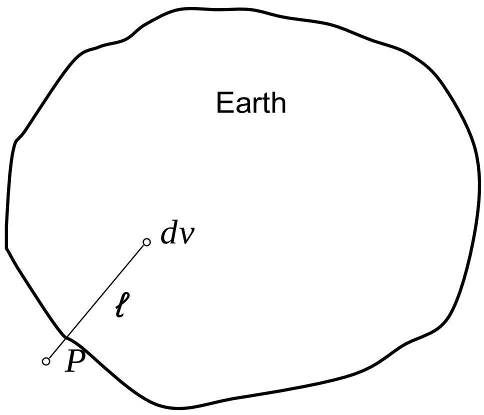

Fig.7. Illustrating the gravitational potential

For the non-mathematician, this formula looks dangerous indeed, but it is conceptually simple. Its basic structure is

$$V = L\,\rho,$$

meaning that the potential *V* is obtained by simply applying the "linear operator" *L* to the density ρ. When I was a student, the word "operator" would have sounded rather dangerous, like a surgical operation. Now, in the age of computers, this word is used for almost anything, such as addition or subtraction and any other "operation" that can be performed by a computer, such as numerical integration.

So far, everything is simple and classical: no relativity, no complexity, no chaos and what not. It can be computed by standard formulas for numerical integration, which is easy if ρ is given and *V* is to be determined.

The **inverse problem**, to compute the density ρ if the potential *V* is given, is extremely difficult, since there are infinitely many possible mass distributions that generate a given external gravity field. This inverse problem is of great importance for geophysical exploration for oil and minerals. It is easy to write the solution in the symbolic form

[10] For instance: B. Hofmann-Wellenhof and H. Moritz, Physical Geodesy, Springer, Wien, 2005.

$$\rho = L^{-1} V$$

but the actual calculation raises many problems.

Without going into details, such inverse problems are called *improperly posed* or *ill-posed problems* without any negative connotation. They may be extremely difficult but pose fascinating challenges. This is nowadays an important and active field of mathematical research.[11] (An *improperly posed* or *well-posed problem* according to Hadamard has one and only one solution which is stable; formerly it was mistakenly thought that all physically meaningful problems were well-posed.)

For mathematicians familiar with matrix calculus, there is a simple instructive example. Square matrices with nonzero determinant have a regular inverse matrix; this is well-posed problem. It has, however, turned out useful to define a generalized inverse even for *rectangular* matrices. They are usually ill-posed in the sense of Hadamard.

10. Other inverse problems

Even students in secondary schools must learn how to differentiate and integrate. They know that every analytical function can be differentiated quite easily, but the *inverse problem*, integration is frequently not so easy, requiring integral tables or computer program packages such as Mathematica© (which I used to compute Fig. 3).

A very general problem, the determination of some physical functional object by indirect measurements, is usually ill-posed, because the functional object usually has an infinite number of defining parameters and the number of measurements is always only finite, not to speak about measuring errors. This is the classical problem of *weather prediction*, which we have mentioned above in sec.4, case (1), and also *gravity field estimation* from satellite data.

In geodesy we have *the geodetic boundary-value problem* formulated by M. S. Molodensky (Section 2). Molodensky's problem has been discovered around 1945, and has revolutionized theoretic thinking in geodesy. He also gave several practically useful solutions. A rigorous proof of existence and uniqueness of the nonlinear Molodensky problem under mathematically precisely defined conditions as a "hard" nonlinear inverse problem (and it is "hard" indeed!) has been given by the well-known Swedish mathematician Lars Hörmander, after preliminary work by Torben Krarup (Copenhagen). Krarup also discovered l*east-squares collocation*; see below.

We also mention *seismic tomography* in geophysics, which is structurally very similar to *magnetic resonance tomography* widely applied in medicine.

If we wish to associate the name of a great mathematician of the past with ill-posed problems, we may well take the "princeps mathematicorum", Carl Friedrich Gauss. His *least-squares adjustment of overdetermined systems* very well fits into the picture. This mathematical model, which is classical in geodesy, is defined by p parameters, to be determined by $n > p$ observations. A simple standard example, which Gauss frequently met in his geodetic triangulation work, is the measurement of all three angles of a triangle. Their sum should be precisely 180 degrees, but this never happens in practice because of unavoidable measuring errors. The "solution" of measuring two angles only is biased and unfair to the unmeasured angle. Generally speaking, the use of *all* data gives an optimal

[11] For general reference we mention: G.Anger, *Inverse Problems in Differential Equations,* Akademie Verlag , Berlin, and Plenum Press, London, 1990; G.Anger et al., *Inverse Problems: Principles and Applications in Geophysics, Technology and Medicine,* Akademie Verlag, Berlin 1993 (reprint by Wiley); G. Anger and H. Moritz, Inverse Problems and Stability: Basic Ideas and Applications , in the present website..

unbiased solution and, furthermore, a statistical estimate for the measuring errors involved.

Least- squares collocation, mentioned above, was found by Krarup as least-squares adjustment in Hilbert space.

By the way, the solutions of *linear* Gaussian adjustment problems can be represented in terms of generalized matrix inverses in the sense of Section 9. However, most inverse problems are nonlinear and therefore much more difficult.

11. Conclusions

We conclude with several examples showing the generality of inverse problems, not only in mathematics:

Examples of Direct and Inverse Problems

gravity from mass	mass from gravity
Nature (Earth)	seismic tomography
Nature (human body)	medical tomography (X-ray, MRI, etc.)
Nature	natural science
crime	police
(logical) deduction	(scientific) induction
cause → effect	effect → cause
reality → data	data → reality

A two-dimensional photograph is a projection of a three-dimensional landscape onto a plane. In an abstract sense, the set of measurements forms a projection of the "space of nature" which is infinitely-dimensional when taking all its properties into account, whereas the number of measurements is finite (see also secs. 2 and 10). **Any science is a projection of the "space of nature" onto the subspace defined by the science under consideration.** The Table above shows that, if the projection of reality onto our measuring data is a direct problem, then the computation of the underlying reality from the data is the corresponding inverse problem.

This sounds very far from common sense, but this type of terminology is rapidly entering our language, not only of mathematics, but also of science, finding its way into computer language and hence to our daily way of speech, not only in mathematics, but also in other sciences such as history, as the Table shows. The criminal performs a direct problem: his crime; the detective has the inverse problem to clarify the crime and to find the criminal. Of course, this is much more difficult. The historian looks for the complex reasons and people having produced the present situation: an inverse problem, too.

In fact, general terminology looks very simple because it uses, by analogy, concepts familiar from ordinary speech, avoiding all the embarrassing details. It is easy to write $V = L\rho$ for the "direct problem" and $\rho = L^{-1} V$ for the corresponding inverse problem. We can drive a car without knowing the detailed working of engine, transmission, and brakes, which may be very "complex" indeed.

COMPLEXITY AND INVERSE PROBLEMS frequently work together, in particular with huge data sets. We illustrate this by means of our favourite example, *weather prediction*. The well-known uncertainty of this problem comes from two factors:

- The "chaotic" instability of its differential equations even in the presence of perfect data (the *Lorenz butterfly* phenomenon discussed in Part B of this paper) and
- the lack of the meteorological measurements in the data grid, which should be given in a uniform net densely covering the whole earth's surface. This is an ill-posed problem in the sense just discussed.

This is a geophysical example which shows clearly how the new "nonclassical" problems and methods in natural sciences, discussed in Parts B and C of this paper, play an increasing role in addition to the "classical" methods of Part A. It is clear that the solution of these complex new problems presupposes the powerful contemporary (and future) computers. Still, it even more also presupposes the reasoning and intuition of the human mind.

I believe that complexity and inverse problems will play in future an important interdisciplinary role for many sciences: for physics, chemistry, technology, for biology and, of course, for geosciences.